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Mielnik and Cantoni Transition Probabilities

Sylvia Pulmannová[1]

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It is shown that when a Mielnik transition probability space is given, Cantoni transition probabilities can also be defined on it. A condition is given under which these transition probabilities are equal.

1. INTRODUCTION

By a transition probability space (tps) we mean a couple (S, p) where S is an abstract set and p: $S\times S\to[0, 1]$ satisfies the following conditions:

(i) $p(\alpha, \beta)=1$ iff $\alpha=\beta$.

(ii) $p(\alpha, \beta)=0$ iff $p(\beta, \alpha)=0$.

(iii) Calling α and β orthogonal ($\alpha \perp \beta$) if $p(\alpha, \beta)=0$, we have

$$\sum_{\beta\in R} p(\alpha, \beta)=1$$

for every maximal pairwise orthogonal subset R of S and every $\alpha\in S$.

The set S is interpreted as a set of states of a physical system and $p(\alpha, \beta)$ as the probability of transition from the state α to the state β. The notion of an abstract transition probability space was introduced by Mielnik (1968). We note that our definition is more general than Mielnik's original definition: we replaced the symmetry condition $p(\alpha, \beta)=p(\beta, \alpha)$ by the weaker condition (ii).

It has been shown that to every tps an orthomodular structure (a quantum logic) is related (Belinfante, 1976; Deliyannis, 1984; Pulmannová, 1986). Recall that a quantum logic is a partially ordered set L with 0 and 1, with the orthocomplementation $':L\to L$ such that:

(i) $(a')'=a$.

[1]Mathematics Institute, Slovak Academy of Sciences, 814 73 Bratislava, Czechoslovakia.

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(ii) $a \leq b$ implies $b' \leq a'$.

(iii) $a \vee a' = 1$.

(iv) Calling $a, b \in L$ orthogonal ($a \perp b$), if $a \leq b'$, the supremum $\bigvee a_i$ exists in L for any sequence $\{a_i\}$ of pairwise orthogonal elements of L (i.e., L is σ-orthocomplete).

(v) $a \leq b$ implies that there is $c \in L$ such that $c \perp a$ and $a \vee c = b$ (i.e., L is orthomodular).

A state on quantum logic L is a map $m: L \to [0, 1]$ such that (i) $m(1) = 1$ and (ii) $m(\bigvee a_i) = \Sigma\, m(a_i)$ for any sequence $\{a_i\}$ of pairwise orthogonal elements of L. That is, a state is a σ-additive probability measure on L.

A logic L is called orthocomplete if $\bigvee a_i$ exists in L for any set of pairwise orthogonal elements of L. A state m on L is called completely additive if $m(\bigvee a_i) = \sum m(a_i)$ for any set $\{a_i\}$ of pairwise orthogonal elements of L such that $\bigvee a_i$ exists in L.

A set M of states on L is *ordering* if $a \not\leq b$ implies that there is $m \in M$ such that $m(a) > m(b)$, and *strongly ordering* if $a \not\leq b$ implies that there is $m \in M$ such that $m(a) = 1$ and $m(b) \neq 1$.

A functional quantum logic is a set $L \subset [0, 1]^M$, where M is any set, which satisfies the following conditions:

(i) $1 \in L$, where $1(x) = 1$ for all $x \in M$.

(ii) $f \in L$ implies $1 - f \in L$.

(iii) With f and g called orthogonal ($f \perp g$) if $f + g \leq 1$ [i.e., $f(x) + g(x) \leq 1$ for all $x \in M$], we have $\sum f_i \in L$ for any sequence $\{f_i\}$ of pairwise orthogonal elements of L.

The set $(L, \leq, ', 0, 1)$, where $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in M$ and $f' = 1 - f$, is a quantum logic. Moreover, every $x \in M$ generates a state m_x on L by the prescription $m_x(f) = f(x)$, and the set $\{m_x | m \in M\}$ is ordering for L (Maczyński, 1974). The set $\{m_x | x \in M\}$ is strongly ordering for L if $\{x | f(x) = 1\} \subset \{x | g(x) = 1\}$ implies $f \leq g$. A functional quantum logic is orthocomplete if the condition (iii) is replaced by:

(iii') $\sum f_i \in L$ for any set $\{f_i\}$ of pairwise orthogonal elements of L.

2. CANTONI TRANSITION PROBABILITIES ON A TPS

An observable on a quantum logic L is a map x from the Borel subsets $B(\mathcal{R})$ of the real line $\mathcal{R}$ to L such that:

(i) $x(\mathcal{R}) = 1$.

(ii) $x(E^c) = x(E)'$, where $E^c = \mathcal{R} - E$.

(iii) $x(\bigcup E_i) = \bigvee x(E_i)$ for any sequence $\{E_i\}$ of disjoint elements of $B(\mathcal{R})$.

If m is a state on L, then $m_x = m \circ x$ is a probability measure on $B(\mathcal{R})$. Let x be an observable and p, q be states on L. Then there is a finite Borel measure σ such that $p_x, q_x \ll \sigma$ and the expression

$$T_x^{1/2}(p, q) = \int \left(\frac{dp_x}{d\sigma}\right)^{1/2} \left(\frac{dq_x}{d\sigma}\right)^{1/2} d\sigma$$

is independent of σ. Following Cantoni (1975); we define

$$T(p, q) = \inf_x T_x(p, q)$$

where the infimum is over all observables on L. The $T(p, q)$ is the Cantoni transition probability for p, q.

Let (S, p) be a tps and let L be the corresponding quantum logic. To sketch briefly the construction of L, we shall follow Pulmannová (1986). Let us denote by $\mathscr{B}$ the set of all pairwise orthogonal subsets B of S. We suppose that $\varnothing \in \mathscr{B}$ and $\{\alpha\} \in \mathscr{B}$ for any $\alpha \in S$. By Zorn's lemma, to every $B \in \mathscr{B}$ there is a $C \in \mathscr{B}$ such that $B \cup C$ is a maximal subset of pairwise orthogonal elements of S (a base of S). For B_1, B_2 in $\mathscr{B}$ we put $B_1 \sim B_2$ if there is $C \in \mathscr{B}$ such that $B_1 \cup C$ and $B_2 \cup C$ are bases of S. Then $\sim$ is an equivalence relation. Let $\tilde{\mathscr{B}}$ be the set of all equivalence classes, i.e., $\tilde{\mathscr{B}} = \mathscr{B}/\sim$. Let $\tilde{B}$ denote the equivalence class containing B. We put

$$f_{\tilde{B}}(\alpha) = \sum_{\beta \in B} p(\alpha, \beta)$$

It can be shown that $f_{\tilde{B}}$ is well defined (i.e., independent of the choice of the representant B of $\tilde{B}$), and the set $L = \{f_{\tilde{B}} \mid \tilde{B} \in \tilde{\mathscr{B}}\}$ forms an orthocomplete atomistic quantum logic with the atoms f_α, where α is the unique representant of the class $\tilde{\alpha}$. We note that the atomicity of L is implied by the conditions (i) and (iii) of the definition of tps. These conditions also imply that the states m_α on L, defined by

$$m_\alpha(f_{\tilde{B}}) = \sum_{\beta \in B} p(\alpha, \beta)$$

must be pure (i.e., m_α cannot be represented as a convex combination of any other states m_β, m_γ, β, $\gamma \in S$). As a consequence of (iii), we get that L is orthocomplete and the states m_α, $\alpha \in S$, are completely additive.

To simplify the notations, we shall write α instead of m_α to denote the state on L generated by the element $\alpha \in S$. To compare the Cantoni transition probabilities $T(\alpha, \beta)$ with the original transition probabilities $p(\alpha, \beta)$, we shall use the Gudder (1981) expression

$$T(\alpha, \beta)^{1/2} = \inf \sum \alpha(a_i)^{1/2} \beta(a_i)^{1/2}$$

where the infimum is over all finite maximal orthogonal sequences in L (an orthogonal sequence $\{a_1, a_2, \ldots, a_n\}$ is maximal if $\bigvee_{i=1}^n a_i = 1$).

Theorem 1. Let (S, p) be a tps. Then

$$T(\alpha, \beta)^{1/2} = \inf_R \sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

where the infimum is over all bases R of S.

Proof. By atomicity and orthomodularity of L, every $a \in L$ can be expressed as a supremum of pairwise orthogonal atoms. Let $\{a_i\}_{i \le n}$ be a finite maximal orthogonal sequence in L. Let $a_i = \bigvee_j b_{ij}$, $i = 1, 2, \ldots, n$, where $\{b_{ij}\}_j$ are sets of mutually orthogonal atoms in L. By the Schwarz inequality,

$$\sum_{i=1}^{n} \alpha(a_i)^{1/2} \beta(a_i)^{1/2} = \sum_{i=1}^{n} \left[\sum_j \alpha(b_{ij})\right]^{1/2} \left[\sum_j \beta(b_{ij})\right]^{1/2}$$
$$\ge \sum_i \sum_j \alpha(b_{ij})^{1/2} \beta(b_{ij})^{1/2} = \sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

where $R = \{b_{ij})_{i,j}$ is a base of S. [Recall that atoms in L are of the form f_γ and $\alpha(f_\gamma) = p(\alpha, \gamma)$.] This implies that

$$T(\alpha, \beta)^{1/2} = \inf \sum \alpha(a_i)^{1/2} \beta(a_i)^{1/2}$$
$$\ge \inf_R \sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

where the first infimum is over all finite maximal orthogonal sequences $\{a_i\}$ in L and the second infimum is over all bases R of S.

For every base R of S we have

$$\sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2} = \sup_K \sum_{\gamma \in K} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

where the supremum is over all finite subsets K of R. Therefore, there is a sequence $\{K_n\}$ of finite subsets of R such that

$$\sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2} = \lim_{n \to \infty} \sum_{\gamma \in K_n} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

For every K_n,

$$\{f_\gamma | \gamma \in K_n\} \cup \left\{ \bigvee_{\gamma \in R - K_n} f_\gamma \right\}$$

is a finite maximal orthogonal sequence in L. We have

$$\sum_{\gamma \in K_n} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2} + \left[\sum_{\gamma \in R - K_n} p(\alpha, \gamma)\right]^{1/2}$$
$$\times \left[\sum_{\gamma \in R - K_n} p(\beta, \gamma)\right]^{1/2} \ge \sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

which implies that

$$\lim_{n\to\infty}\left[\sum_{\gamma\in R-K_n} p(\alpha,\gamma)\right]^{1/2}\left[\sum_{\gamma\in R-K_n} p(\beta,\gamma)\right]^{1/2}=0$$

From this it follows that for every base R,

$$\sum_{\gamma\in R} p(\alpha,\gamma)^{1/2}p(\beta,\gamma)^{1/2}=\inf\sum \alpha(a_i)^{1/2}\beta(a_i)^{1/2}$$

where the infimum is over all finite maximal sequences $\{a_i\}$ such that every a_i is the supremum of some f_γ, $\gamma\in R$. This, together with the first part of the proof, yields the required equality. ■

Corollary 1. $T(\alpha,\beta)$ has the following properties:
(i) $T(\alpha,\beta)\leq 1$, $T(\alpha,\beta)=1$ iff $\alpha=\beta$.
(ii) $T(\alpha,\beta)=T(\beta,\alpha)$.
(iii) $T(\alpha,\beta)\leq\min\{p(\alpha,\beta),p(\beta,\alpha)\}$.

Proof. (i)

$$\begin{aligned}T(\alpha,\beta)^{1/2}&=\inf_R\sum_{\gamma\in R} p(\alpha,\gamma)^{1/2}p(\beta,\gamma)^{1/2}\\&\leq\inf_R\left[\sum_{\gamma\in R} p(\alpha,\gamma)\right]^{1/2}\left[\sum_{\gamma\in R} p(\beta,\gamma)\right]^{1/2}=1\end{aligned}$$

If $\alpha=\beta$, then

$$T(\alpha,\alpha)^{1/2}=\inf_R\sum_{\gamma\in R} p(\alpha,\gamma)=1$$

On the other hand, let

$$\inf_R\sum_{\gamma\in R} p(\alpha,\gamma)^{1/2}p(\beta,\gamma)^{1/2}=1$$

Take R such that $\alpha\in R$; then $p(\alpha,\gamma)=0$ for $\gamma\in R-\alpha$. Therefore

$$1=\sum_{\gamma\in R} p(\alpha,\gamma)^{1/2}p(\beta,\gamma)^{1/2}=p(\beta,\alpha)^{1/2}$$

which implies $\beta=\alpha$.

(ii) is immediate.

Note that (i) and (ii) are satisfied by any Cantoni transition probabilities.

(iii) Take R such that $\beta\in R$. Then

$$\sum_{\gamma\in R} p(\alpha,\gamma)^{1/2}p(\beta,\gamma)^{1/2}=p(\alpha,\beta)^{1/2}\geq T(\alpha,\beta)^{1/2}$$

Similarly, taking R such that $\alpha\in R$, we obtain $p(\beta,\alpha)\geq T(\alpha,\beta)$.

Corollary 2. We have $T(\alpha, \beta) = p(\alpha, \beta)$ iff for every base R of S the following inequality holds:

$$p(\alpha, \beta)^{1/2} \leq \sum_{\gamma \in R} p(\alpha, \gamma)^{1/2} p(\beta, \gamma)^{1/2}$$

If α, β are unit vectors in a Hilbert space H and $p(\alpha, \beta) = |\langle \alpha, \beta \rangle|^2$, where $\langle \cdot, \cdot \rangle$ is the inner product in H, then for every orthonormal base R in H we have

$$|\langle \alpha, \beta \rangle| = \left| \sum_{\gamma \in R} \langle \alpha, \gamma \rangle \langle \gamma, \beta \rangle \right| \leq \sum_{\gamma \in R} |\langle \alpha, \gamma \rangle| \, |\langle \beta, \gamma \rangle|$$

so that the condition of Corollary 2 is satisfied [compare with Hadjisavvas (1982)]. The condition of Corollary 2 can be considered as a necessary condition of the embeddability of a tps into a Hilbert space. In particular, this condition implies the symmetry of p. The following example (Belinfante, 1976) shows that the condition of Corollary 2 need not be satisfied even in a symmetric tps:

	α	β	γ	δ	π	ρ	ε	η
α	1	0	0	a	$1-a$	a	0	0
β	0	1	0	$1-a$	a	$1-a-b$	b	$1-b$
γ	0	0	1	0	0	b	$1-b$	b
δ	a	$1-a$	0	1	0	$1-b$	b	$1-a-b$
π	$1-a$	a	0	0	1	b	0	a
ρ	a	$1-a-b$	b	$1-b$	0	1	0	$1-a$
ε	0	b	$1-b$	b	0	0	1	0
η	0	$1-b$	b	$1-a-b$	a	$1-a$	0	1

There are four bases, (α, β, γ), $(\alpha, \varepsilon, \eta)$, (γ, δ, π), (π, ρ, ε). An easy computation shows that $T(\beta, \varepsilon) = (1-a)b$, while $p(\beta, \varepsilon) = b$.

Note that if the tps is generated by a (total) transition amplitude space (Gudder and Pulmannová, 1987), the condition of Corollary 2 is satisfied.

Let $f_\alpha \in L$ be defined by $f_\alpha(\beta) = p(\beta, \alpha)$, $\beta \in S$. Let x_α be the observable defined by

$$x_\alpha(E) = \begin{cases} f_\alpha & \text{if} \quad 1 \in E, \quad 0 \notin E \\ 1 - f_\alpha & \text{if} \quad 1 \notin E, \quad 0 \in E \\ 0 & \text{if} \quad 1 \notin E, \quad 0 \notin E \\ 1 & \text{if} \quad 1 \in E, \quad 0 \in E \end{cases}$$

Proposition 1. Let (S, p) be a tps. Then for every $\alpha, \beta \in S$, $p(\beta, \alpha) = T_{x_\alpha}(\alpha, \beta)$.

Proof. For any observable x we have

$$T_x(\alpha, \beta)^{1/2} = \inf \sum_i \alpha_x(E_i)^{1/2} \beta_x(E_i)^{1/2}$$

where the infimum is over all finite Borel partitions (Gudder, 1981). For any finite Borel partition which is fine enough to separate 0 and 1, we have

$$\sum_i \alpha_{x_\alpha}(E_i)^{1/2}\beta_{x_\alpha}(E_i)^{1/2} = \alpha(f_\alpha)^{1/2}\beta(f_\alpha)^{1/2} + \alpha(1-f_\alpha)^{1/2}\beta(1-f_\alpha)^{1/2}$$
$$= p(\beta, \alpha)^{1/2}$$

Therefore, $T_{x_\alpha}(\alpha, \beta) = p(\beta, \alpha)$. ■

3. METRICS ON A TPS

Gudder (1981) has shown that

$$d_1(\alpha, \beta) = \{2[1 - T(\alpha, \beta)^{1/2}]\}^{1/2}$$

is a metric on the states of a logic L. In the case of a tps logic, d_1 is a metric on S. By Cantoni (1985),

$$d_2(\alpha, \beta) = 2 \arccos[T(\alpha, \beta)^{1/2}]$$

is also a metric on S. It is easily seen that the topologies induced by d_1 and d_2 are equivalent.

For a subset M of S put $M^\perp = \{\alpha \in S | p(\alpha, \beta) = 0 \text{ for all } \beta \in M\}$.

Proposition 2. Let (S, p) be a tps satisfying the condition of Corollary 2. If $M \subset S$ is such that $M = M^{\perp\perp}$, then M is closed in the topology induced by $d = d_1$ (or $d = d_2$).

Proof. Let $\alpha_n \in M$, $d(\alpha_n, \alpha) \to 0$. Let $\beta \in M^\perp$. Then $p(\alpha_n, \beta) = 0$, which implies $T(\alpha_n, \beta) = 0$, $n = 1, 2, \ldots$. Therefore $d(\alpha, \alpha_n) + d(\beta, \alpha) \geq d(\alpha_n, \beta) = \sqrt{2}$. Hence $d(\alpha, \beta) = \sqrt{2}$, which implies $T(\alpha, \beta) = p(\alpha, \beta) = 0$, hence $\alpha \in M^{\perp\perp} = M$.

We say that a subset M of S is a subspace if for any finite subset F of M we have $F^{\perp\perp} \subset M$. Proposition 2 implies that provided the condition of Corollary 2 is satisfied, every finite subspace $F^{\perp\perp}$ is topologically closed. It is an open question under what conditions on a tps the equality $\bar{M}^d = M^{\perp\perp}$ holds for every subspace M of S. ■

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